

Longest Path in Percolating Hierarchical Lattice

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We determine numerically the probability distribution for the longest self-avoiding path lengths connecting two distant points on a diluted hierarchical lattice at the percolation threshold. The evolution of this distribution with the system size is studied and the distribution is observed to approach a universal scale-invariant form under proper rescaling of its argument. The longest path length scales as $|\Delta p|^{-\zeta_{\max}}$ and our estimate for $\zeta_{\max} = 1.816 \pm 0.013$ is clearly different from the previously estimated $\zeta_{\min} = 1.531 \pm 0.002$ for the shortest path lengths on the same hierarchical lattice. This gives support to the multifractal behavior of SAWs on percolating clusters.

KEY WORDS: Self-avoiding walk; percolation threshold; critical exponents; multifractal behavior.

1. INTRODUCTION

In a dilute lattice, the bonds/sites of the lattice are removed at random with a finite probability $1 - p$, and a self-avoiding walk (SAW) on such a lattice has to avoid the vacant sites or bonds. In recent years, there has been a number of studies⁽¹⁾ on the effect of the lattice dilution to the SAW behavior, especially at the percolation threshold p_c of the system. It has been pointed out⁽²⁾ that on a percolation cluster, different self-avoiding connections between any two widely separated points (at a distance of the order of the correlation length ξ) scale with different exponents, forming a continuous spectrum and one needs an infinite set of exponents to fully characterize the probability distribution of the SAW lengths. Thus, the shortest SAW L_{\min} , the longest SAW L_{\max} , and the average SAW length $\langle L \rangle$ between any pairs of points all have different scaling exponents at p_c .

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While the behavior of L_{\min} or $\langle L \rangle$ has been studied in some details,³ the longest path L_{\max} has not received the same attention. In this paper, we study L_{\max} , precisely the length of the longest SAW that on average can be embedded on a percolating lattice, and how this length behaves at p_c .

Our calculations are performed on a diluted Wheatstone-bridge hierarchical lattice (Fig. 1) described in the next section. We focus our attention on the probability distribution $P_s(l)$ of the longest SAW lengths l in a system of size s , and especially on the manner in which this distribution evolves as the system size s is increased. The distribution is determined numerically for $p = p_c$. At p_c , the distributions $P_s(l)$ for different system size s are found to approach a nontrivial function under proper rescaling of the arguments. The function $\tilde{P}(x)$ which the scaled distribution attains is universal, as are the critical exponents. We obtain the exponent ζ_{\max} , which describes the scaling of the length of the longest SAW, $L_{\max} \sim \xi^{\zeta_{\max}/\nu} \sim |\Delta p|^{-\zeta_{\max}}$, where ν is the correlation length exponent. Our estimate of $\zeta_{\max} = 1.816 \pm 0.013$ is much greater than $\zeta_{\min} = 1.531 \pm 0.002$ for the shortest path (see ref. 3) on such a hierarchical lattice, indicating the multifractal behavior of SAWs on a percolating cluster. Though our calculations have been done on a hierarchical lattice, we expect that this result would be qualitatively valid on regular lattices as well.

2. THE MODEL AND ENUMERATION PROCESS

The hierarchical lattice we consider is of the Wheatstone bridge type as shown in Fig. 1. At the zeroth order ($n=0$) we have only a primitive bond as shown in Fig. 1a. In the next stage, five such primitive bonds are assembled to form the unit in the first order ($n=1$) as shown in Fig. 1b, and the process is carried on repetitively to form the lattice at any arbitrary order n . We consider the bond percolation problem on such a hierarchical lattice where each bond of the lattice is present with a probability p . At any order n , we determine the probability $P(l)$ for the longest self-avoiding path l ($2^n \leq l \leq 3^n$) that connects the topmost point T_n to the lowest point B_n (see Fig. 1) through the set of occupied bonds. The enumeration process closely follows the one used for the determination of shortest paths on such hierarchical lattice in Barma and Ray.⁽³⁾ The primitive bond ($n=0$) in Fig. 1a is present with probability p and has $l=1$; otherwise, when the bond is absent (this has the probability $1-p$), we have $l=0$. At $n=1$

³ For the study of L_{\min} ($\sim \xi^{\zeta_{\min}/\nu}$, where ν is the percolation correlation length exponent) on a percolation cluster, see Barma and Ray⁽³⁾ and the references therein, and for a recent review on the behavior of $\langle L \rangle$ ($\sim \xi^{1/\nu_{\text{SAW}}}$, where ν_{SAW} is the random SAW size exponent on the percolation cluster) in the presence of dilution see ref. 1.

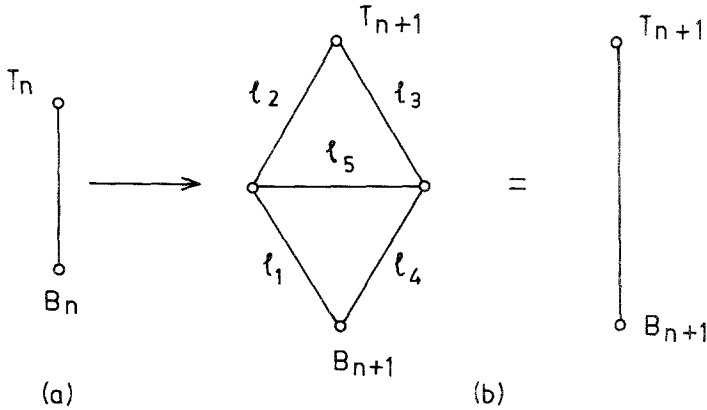


Fig. 1. The generation of the hierarchical lattice from the n th to the $(n+1)$ th level. The longest path connecting T_{n+1} and B_{n+1} is calculated from the five n -level longest paths l_1, l_2, \dots, l_5 .

each of the l_i ($i = 1, 2, \dots, 5$) in Fig. 1b is either 1 or 0 as in the case of the $n = 0$ level bond and the maximum path lengths assume the values 3, 2, or 0 according to the probabilities $P_1(3) = -p^5 + 2p^3$, $P_1(2) = 3p^5 - 5p^4 + 2p^2$, or $P_1(0) = 1 - P_1(2) - P_1(3)$, respectively. It is clear that for any n the longest SAW length l connecting T_n and B_n is a function $f(l_1, l_2, \dots, l_5)$ only of the longest SAW lengths l_1, l_2, \dots, l_5 that appear in the $(n-1)$ th order. Explicitly,

$$f(l_1, l_2, \dots, l_5) = \text{Max}[l_1 + l_2, l_4 + l_3, l_1 + l_5 + l_3, l_4 + l_5 + l_2] \quad (1)$$

with the sum rule $\sum l_i = 0$ if any of the components $l_i = 0$. The probability distribution $P_n(l)$ is then determined by

$$P_n(l) = \sum_{l_1 \dots l_5} \sum_{i=1}^5 P_{n-1}(l_i) \delta(l - f(l_1, l_2, \dots, l_5)) \quad (2)$$

where the finite sums over the l_i ($i = 1, \dots, 5$) run through all the integral values from 2^{n-1} to 3^{n-1} .

We use a Monte Carlo method to study the evolution of $P_n(l)$. A random number generator is used to select each l_i with relative frequency $P_{n-1}(l)$. For each such try, l is determined using Eq. (1) and the distribution $P_n(l)$ is obtained by repeating the procedure many times (~ 200 times for $n = 2$ and $\sim 4000n$ times for higher n) for different $\{l_i\}$ configurations.

We start from $n = 1$, for which we know l and the distribution $P_1(l)$ exactly, and use the Monte Carlo method to generate l and $P_n(l)$ for $n = 2$ and onward (we have been able to compute up to $n = 10$). For finite n the

distribution $P_n(l)$ consists of a series of delta functions at the integral values of l from 2^n to 3^n and the sum $\sum P_n(l) = p_n$, the probability that T_n and B_n are connected at the n th order of construction. The distance between T_n and B_n increases with n and the evolution of $P_n(l)$ as $n \rightarrow \infty$ becomes important.

3. SCALING FORM AT $p = p_c$

The hierarchical lattice we have considered has $p_c = 0.5$ exactly for all n . At any generation n , if p_n is the effective probability that the nodal points T_n and B_n are connected, then p_{n+1} is determined in terms of p_n as

$$p_{n+1} = 2p_n^5 - 5p_n^4 + 2p_n^3 + 2p_n^2 \tag{3}$$

This recursion relation, true for any n , is exact in this lattice and has an unstable fixed point $p_c = 0.5$.⁽⁴⁾ At the n th generation, T and B are separated by 2^n bonds, and at the $(n + 1)$ th generation by 2^{n+1} bonds, so that the scale factor $b = 2$. The correlation length exponent ν is obtained on linearizing Eq. (3) around p_c as

$$\nu = \ln b / \ln(\delta p_{n+1} / \delta p_n)|_{p_c} = 1.428$$

At p_c , the probability distribution $P(l)$ at any n is expected to approach a scale-invariant form

$$P(l) = \lambda_n^{-1} \tilde{P}(l/\lambda_n) \tag{4}$$

where the scale factor is

$$\lambda_n = \chi^n \tag{5}$$

We estimate χ from the mean and the root mean squared longest path at the n th level as follows

$$\begin{aligned} \langle l \rangle_n &\cong c_1 \lambda_n \\ (\langle l^2 \rangle_n)^{1/2} &\cong c_2 \lambda_n \end{aligned} \tag{6}$$

where c_1 and c_2 are constants involving moments of the scaled function \tilde{P} as

$$c_1 = \int dx x \tilde{P}(x), \quad c_2^2 = \int dx x^2 \tilde{P}(x) \tag{7}$$

where $x = l/\chi^n$ for the paths l at the n th generation. From Eqs. (5) and (6), both $\langle l \rangle_{n+1} / \langle l \rangle_n$ and $(\langle l^2 \rangle_{n+1} / \langle l^2 \rangle_n)^{1/2}$ should approach χ for large n .

We have determined $\langle l \rangle_n$ and $\langle l^2 \rangle_n$ from our numerically obtained distributions $P_n(l)$. The ratios versus $1/n$ are plotted in Fig. 2. It appears from the figure that the extrapolated value of χ for $1/n \rightarrow 0$ lies near 2.420. Hence, by putting different values of χ around 2.420 in Eq. (4), it was found that the distributions $P_n(l)$ for $n=7, 8, 9$, and 10 obey the best scaling form for $\chi = 2.415 \pm 0.015$.

The universal form of $\tilde{P}(x)$ as determined from our data for $n=7, 8, 9$, and 10 and $\chi=2.415$ is shown in Fig. 3. Since the number of data points for higher values of n becomes unmanageably large ($\sim 3^n$), we have adopted a binning procedure where $P(l)$ is obtained on averaging $P(l_i)$ over a finite range Δ of l values so that the final $P(l) = 1/\Delta \{ \sum_{l_i}^{l_1+\Delta} P(l_i) \}$, and $l = l_1 + \Delta/2$. The value of Δ has been chosen depending on n ; as n increases, Δ is increased. Typically, $\Delta \sim 25$ for $n=7$ and $\Delta \sim 100$ for $n=9$ and above. Hence, Fig. 3 shows a finite number of points.

The average longest path $\langle l \rangle_n$ at the n th level depends on its scale was

$$\langle l \rangle_n \sim R_n^{d_s} \tag{8}$$

where R_n is the distance between T_n and B_n and can be taken as $R_n = 2^n$ as the basic scale. From Eqs. (6) and (8) we get for large n

$$d_s = \ln \chi / \ln 2 = 1.272 \pm 0.009 \tag{9}$$

and for $R_n \sim$ the correlation length ξ ,

$$L_{\max} \sim \xi^{d_s} = |p - p_c|^{-\nu d_s} \tag{10}$$

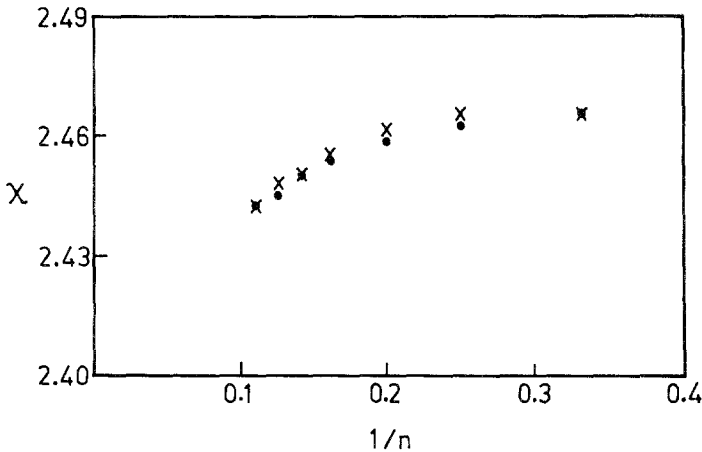


Fig. 2. The ratios of (●) the mean and (×) root mean squared longest path lengths plotted against $1/n$. The χ is estimated from the value of these ratios as $1/n \rightarrow 0$.

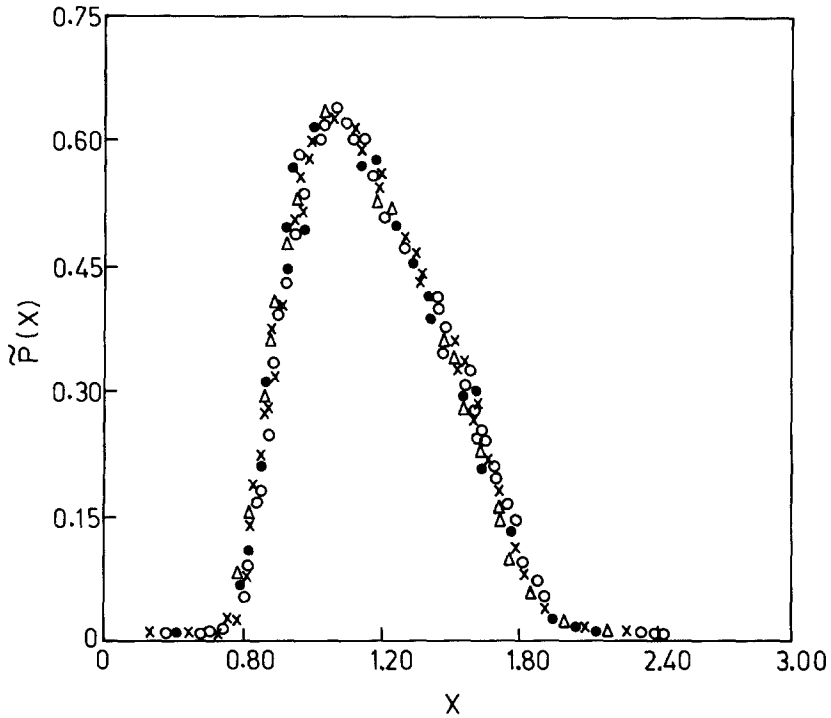


Fig. 3. The scaled probability distribution $\tilde{P}(x)$ with $x=l/\chi^n$ at p_c for values of $n=(\Delta)$ 7, (\bullet) 8, (\circ) 9, and (\times) 10, for $\chi=2.415$.

Hence the longest path exponent ζ_{\max} turns out to be

$$\zeta_{\max} = \nu d_s = 1.816 \pm 0.013 \quad (11)$$

The error estimates for d_s and hence ζ_{\max} correspond to that of χ found from Fig. 2.

4. DISCUSSION

In this paper, we have determined the length of the longest self-avoiding path L_{\max} between any two points in a percolating system and its variation as the distance between the points is increased. Calculations are performed on a bond diluted Wheatstone bridge hierarchical lattice and the entire probability distribution $P_n(l)$ of the length l of the longest SAWs at p_c is determined numerically for different generations ($n=1-10$) of the lattice. $P_n(l)$ is observed to approach a scale-invariant form $\tilde{P}(l/\lambda_n)=$

$\lambda_n P_n(l)$ with the scale factor $\lambda_n = \chi^n$, where χ is found to be equal to 2.415 ± 0.015 . The scaling form of L_{\max} is obtained as $L_{\max} \sim |\Delta p|^{-\zeta_{\max}}$ with $\zeta_{\max} = 1.816 \pm 0.013$.

The result for the hierarchical lattice can be expected to be valid only qualitatively for regular lattices. However, the enumeration process on this lattice has some advantages. In fact, extensive work has been done on hierarchical lattices⁽⁵⁾ providing interesting examples of phase transitions and critical behavior. The position space renormalization group (PSRG) transformations as given by Eq. (3) are exact⁽⁴⁾ on this lattice, but are only approximate for regular lattices.⁽⁶⁾ For any generation n , $p_c = 0.5$ exactly, and the scaling form of $P_n(l)$ as given by Eq. (4) can be obtained accurately. On this hierarchical lattice, any SAW at order n can be decomposed into SAWs on sublattices of order $n-1$ and so on, so that knowing the l for small n , one can obtain the corresponding values for larger n and the enumeration problem becomes essentially local. If E_n is the number of SAWs to be checked to find the longest one at any order n , then $E_n = 5E_{n-1} + 4 = 5^n - 1 = 2(N_n - 2)$, where $N_n = (5^n + 3)/2$ is the number of sites in the lattice at that order. Hence the problem is tractable in the sense that the computer time required to find the longest SAW varies linearly with the number of sites in the system. In regular lattices, the problem of finding the longest SAW is essentially global, rendering it much more difficult to check any plausible scaling form.

As expected, our estimate of $\zeta_{\max} = 1.816 \pm 0.013$ is very close to the PSRG estimate $\zeta_{\max} = 1.835$ for two-dimensional regular lattices.⁽⁴⁾ It is also within the error bars of the theoretical estimate $\zeta_{\max} = 1.77 \pm 0.03$ obtained assuming a functional form of $\zeta^{(2)}$. However, it differs slightly from the numerical estimate $\zeta_{\max} \cong 1.59$ on a regular square lattice.⁽²⁾

The value of ζ_{\max} is different from the shortest path exponent $\zeta_{\min} = 1.531 \pm 0.002$ on the same hierarchical lattice,⁽³⁾ showing that the longest and shortest paths scale with different exponents on a percolation cluster. This indicates that there might be a multifractality associated with the scaling of the SAW at p_c . The two values ζ_{\min} and ζ_{\max} might provide bounds for the size exponent v_{SAW} for the mean SAW $\langle L \rangle$ as $\zeta_{\min}/v \leq 1/v_{\text{SAW}} \leq \zeta_{\max}/v$. Therefore, for this lattice, $0.786 \leq v_{\text{SAW}} \leq 0.933$. However, to our knowledge, no estimate of v_{SAW} on a percolation cluster on this lattice is known.

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